

What we do understand of Colour Confinement

A. Di Giacomo^a

^aDipartimento di Fisica Università di Pisa and INFN
Via Buonarroti 2, 56100 Pisa, Italy

A review is presented of what we understand of colour confinement in QCD. Lattice formulation provides evidence that QCD vacuum is a dual superconductor: the chromo-electric field of a $q\bar{q}$ pair is constrained by dual Meissner effect into a dual Abrikosov flux tube and the static potential energy is proportional to the distance.

1. Introduction

Lattice[1] formulation is a gauge invariant regulator of non abelian gauge theories. Numerical simulations on the lattice produce from first principles regulated correlators of physical quantities.

Simulations can be used to compute quantities involving low energy modes, which are out of reach of perturbation theory, such as weak interaction matrix elements, masses, matrix elements of operators in the light cone expansion. The typical problems encountered in this “phenomenological” use of lattice are the removal of the cut off (renormalization), and limitations in computer power.

Simulations can also be used to test theoretical ideas and to investigate the structure of the theory. An example of investigation of “theoretical” type is the study of the mechanism of confinement. The typical difficulty in this approach is to have good theoretical ideas to test numerically, and possibly to falsify from the first principles.

Apart from confinement itself there are a few fundamental issues at the background of our understanding of QCD. Among them

- 1) The $1/N_c \rightarrow 0$ limit. The conjecture[2] is that the basic properties of a gauge theory, e.g. confinement, are already contained in the limit in which the number of colours N_c goes large, with $g^2 N_c$ fixed. Corrections $\mathcal{O}(1/N_c)$ can be treated as a small perturbation. A consequence of this conjecture is that also quark loops can be viewed as a small perturbation, apart from their effect on the scale of the theory. Indeed apart from the loop with two vertices which is proportional to $g^2 N_f \sim N_f/N_c \sim 1$, and which enters in the β function, loops with more vertices have additional factors $1/N_c$ and are negligible. According to this conjecture also the mechanism of confinement and the corresponding order parameter should then be marginally affected by the presence of quarks.
- 2) Understanding the ground state is also important to understand why perturbation theory works at small distances. Perturbative quantization describes interaction of

quarks and gluons, and the ground state is the Fock vacuum. Quarks and gluons are not observed in nature, and the Fock vacuum is certainly not the ground state. This reflects in the renormalized perturbation expansion as a lack of convergence, even in the sense of asymptotic expansion[3].

2. Confinement in Nature.

Colour is confined in nature. The expected ratio of abundance of quarks n_q to abundance of nucleons n_p is in the standard cosmological model[4]

$$\frac{n_q}{n_p} \simeq 10^{-12} . \quad (1)$$

The experimental upper limit is

$$\frac{n_q}{n_p} \leq 10^{-27} , \quad (2)$$

coming from Millikan like experiments on $\sim 1 g$ of matter.

The estimate (1) is conservative. If we assume no confinement and T is the temperature at which quarks decouple their effective mass m_q is $\sim T$. The reactions

$$q + \bar{q} \rightarrow \text{mesons} \quad q + q \rightarrow \bar{q} + \text{baryons}$$

are esothermic: let σ the corresponding cross section and $\sigma_0 \equiv \lim_{v \rightarrow 0} \sigma v$. Then quarks will decouple when

$$n_q \sigma_0 = G_N^{1/2} T^2 . \quad (3)$$

Since $n_\gamma \sim T^3$ this implies

$$\frac{n_q}{n_\gamma} \sim \frac{G_N^{1/2}}{T \sigma_0} = \frac{10^{-18}}{m_p m_q \sigma_0} , \quad (4)$$

$\sigma_0 \sim m_\pi^{-2}$. The ratio (1) corresponds to $m_q = T = 10 GeV$.

The factor 10^{-15} between the observation and the expectation cannot be explained by fine tuning of a small parameter. Like the experimental limit on the resistivity of a superconductor, it can only be explained in terms of symmetry.

A suggestive idea in that direction is that vacuum is a dual superconductor [5]. The chromoelectric field between a $q \bar{q}$ pair is constrained by dual Meissner effect into an Abrikosov flux tube with energy proportional to the distance.

A relativistic version of the free energy of a superconductor, which is the analog of effective action in field theory, is

$$G = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \varphi)^* (D_\mu \varphi) + V(\varphi) , \quad (5)$$

where

$$D_\mu \varphi = (\partial_\mu - iq A_\mu) \varphi \quad (6)$$

is the covariant derivative and

$$V(\varphi) = \mu^2 \varphi^* \varphi - \frac{\lambda}{2} (\varphi^* \varphi)^2 \quad (7)$$

is the effective potential. μ^2 and λ are functions of the temperature, and $\mu^2(T) > 0$ in the superconducting phase, where the potential has a mexican hat shape.

Putting $\varphi = f e^{iq\theta}$, with $f > 0$ gives $D_\mu \varphi = -iq(A_\mu - \partial_\mu \theta) f e^{iq\theta}$. Under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$

$$\partial_\mu \theta \rightarrow \partial_\mu \theta + \partial_\mu \Lambda \quad (8)$$

and $\tilde{A}_\mu = A_\mu - \partial_\mu \theta$ is gauge invariant. Moreover

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = F_{\mu\nu} \quad (9)$$

and the free energy can be rewritten as

$$G = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} + q^2 f^2 \tilde{A}_\mu \tilde{A}_\mu + \partial_\mu f \partial_\mu f + \mu^2 f^2 - \frac{\lambda}{2} f^4 . \quad (10)$$

The equations of motion are

$$\partial_\mu \tilde{F}_{\mu\nu} + q^2 f^2 \tilde{A}_\nu = 0 , \quad (\partial^2 + \mu^2) f = 2\lambda f^3 \quad (11)$$

A static solution in the gauge $A_0 = 0$ has $\vec{E} = 0$, $f = \tilde{f} \equiv \sqrt{\mu^2/2\lambda}$ and eq.(11) reads

$$\vec{\nabla} \wedge \vec{H} + q^2 \tilde{f}^2 \vec{\tilde{A}} = 0 , \quad (12)$$

$$\nabla^2 \vec{H} - q^2 \tilde{f}^2 \vec{H} = 0 . \quad (13)$$

Eq. (12) means that also in the absence of electric field there is a permanent current, or that $\sigma = \infty$. Eq. (13) is Meissner effect: the field $\vec{\tilde{A}}$ penetrates by a length $(q\tilde{f})^{-1}$. At large distance from the center of a flux tube $\vec{\tilde{A}} = 0$ or

$$e \oint \vec{\tilde{A}} d\vec{x} = e\Phi(B) = 2\pi n , \quad (14)$$

which is the Dirac quantization condition. Abrikosov flux tubes have monopoles at their ends.

Those phenomena are a consequence of symmetry[6]: the order parameter is $f = \langle |\varphi| \rangle$, or the non vanishing v.e.v. of a charged operator. For dual superconductivity the signal of the phase should be the v.e.v. $\langle \mu \rangle$ of an operator carrying magnetic charge.

3. Phenomenology of confinement on the lattice.

Lattice produces evidence for confinement. Wilson loops, defined as parallel transport along square contours in space time, provide the static force between $q\bar{q}$ pairs, in the limit of large T

$$W(R, T) \underset{T \rightarrow \infty}{\simeq} \exp(-\sigma V(R)T) \quad (15)$$

The area law observed in lattice gauge theory[7]

$$W(R, T) \simeq \exp(-\sigma RT) \quad (16)$$

means

$$V(R) = \sigma R \quad (17)$$

or that an infinite amount of energy is needed to pull the two particles at infinite distance from each other. σ is the string tension, related to the slope of Regge trajectories.

Also chromoelectric flux tubes between $q\bar{q}$ pairs are observed, with transverse size ~ 0.5 fm[8,9]. The colour orientation of the chromoelectric field inside them can also be studied[9,10].

Finally the collective modes of the string formed by the flux tube can be analysed[11].

All these facts support the picture of confinement as due to dual superconductivity of vacuum. A microscopic understanding is however needed. In particular monopoles which condense in the vacuum have to be identified.

4. Monopoles

In QED, which is a $U(1)$ gauge theory, magnetic charges are omitted, since they are not observed in nature. As a consequence the general solution of Maxwell's equations can be given in terms of a vector potential A_μ . The field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (18)$$

obeys the equations

$$\partial_\mu F_{\mu\nu} = j_\nu . \quad (19)$$

The absence of magnetic charges identically follows from eq.(18). The dual tensor $F_{\mu\nu}^* \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is, by virtue of eq. (18) identically conserved:

$$\partial_\mu F_{\mu\nu}^* = 0 . \quad (20)$$

Eq. (20) is known as Bianchi identity.

The only way to have a monopole and to preserve Bianchi identity is [12] to introduce a singularity, and consider the monopole as the end point of an infinitely thin solenoid (Dirac string), which can be made invisible if the parallel transport of any charge q around it is trivial or if

$$\exp\left(iq \oint \vec{A} \cdot d\vec{x}\right) = 1 . \quad (21)$$

The line integral is intended on a path which encircles the string and is equal to the magnetic flux, or to 4π the magnetic charge of the monopole. Eq. (21) implies $4\pi qM = 2n\pi$ or $qM = \frac{n}{2}$, which is the celebrated Dirac quantization condition. As a consequence the theory becomes compact.

In non abelian gauge theories, in the familiar multipole expansion the monopole term obeys abelian equation of motion, has a Dirac string and a number of independent abelian magnetic charges which is $N-1$ for the gauge group $SU(N)$ [13]. The 't Hooft-Polyakov[14, 15] monopole of the $SO(3)$ Georgi-Glashow model obeys this classification.

5. Monopoles in QCD

To understand the monopoles in QCD we shall phrase the classification of ref. [13] in the language of ref. [16], or in terms of “abelian projections”. We shall refer to $SU(2)$ for simplicity: extension to $SU(N)$ is trivial[17].

Let $\vec{\phi}(x) \cdot \vec{\sigma}$ be any local operator belonging to the adjoint representation. We define $\hat{\phi}(x) = \frac{\vec{\phi}(x)}{|\vec{\phi}(x)|}$. $\hat{\phi}$ is well defined except at zeros of $\vec{\phi}(x)$. Consider the field strength tensor[14]

$$F_{\mu\nu}(x) = \hat{\phi} \cdot \vec{G}_{\mu\nu}(x) - \frac{1}{g} \hat{\phi}(x) \cdot (D_\mu \hat{\phi}(x) \wedge D_\nu \hat{\phi}(x)) \quad (22)$$

where $\vec{G}_{\mu\nu}(x) = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu$ and $D_\mu = \partial_\mu - g \vec{A}_\mu \wedge$ is the covariant derivative. The coefficient of the second term in eq. (22) is chosen in such a way that the quadratic term $\vec{A}_\mu \wedge \vec{A}_\nu$ cancels with the first term. Both terms are gauge invariant under regular gauge transformations.

A gauge transformation $U(x)$ which brings $\hat{\phi}$ along the 3 axis $\hat{\phi} = (0, 0, 1)$, and diagonalizes $\vec{\phi}(x) \cdot \vec{\sigma}$ is called an abelian projection. After abelian projection

$$F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \quad (23)$$

is an abelian field. This holds in all points where $U(x)$ is regular. Defining the dual tensor $F_{\mu\nu}^*$ as

$$F_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \quad , \quad (24)$$

the magnetic current j_μ^M is defined as

$$\partial_\mu F_{\mu\nu}^* = j_\nu^M \quad (25)$$

and is identically conserved. It is zero except at the singular points of $U(x)$, where monopoles can appear. Thus

$$\partial_\mu j_\nu^M = 0 \quad (26)$$

defines a magnetic $U(1)$ symmetry of the theory. It is not a subgroup of the gauge group since both $F_{\mu\nu}$ and j_μ^M are colour singlet. If the vacuum is not invariant under that $U(1)$ symmetry, monopoles condense like the Cooper pairs and there is dual superconductivity.

Notice that

1. there is a magnetic $U(1)$ symmetry for each operator $\vec{\phi}(x)$ in the adjoint representation.
2. Under the abelian projection $U(x)$, due to singularities, the field strength tensor $\vec{G}_{\mu\nu}$ acquires a singular component

$$\vec{G}_{\mu\nu} \rightarrow U \vec{G}_{\mu\nu} U^{-1} + \vec{G}_{\mu\nu}^{sing} \quad . \quad (27)$$

The regular part can have monopole sources. $\vec{G}_{\mu\nu}^{sing}$ describes Dirac strings starting from the monopoles.

The strategy will then be to detect condensation of different monopole species by measuring with numerical simulations across the deconfining transition a “disorder” parameter, which detects dual superconductivity. The usual order parameter is the Polyakov line. Our disorder parameter will be zero in the deconfined phase, where the order parameter is non zero, and different from zero in the confined phase, where it is zero. The concept of disorder parameter is typical of systems which admit a dual description[18]. They usually have extended structures with non trivial topology (monopoles in QCD), which condense in the disordered phase. In a dual description these structures are described by local fields, and the definition of order and disorder is interchanged. Before giving the results and a few details on how they have been obtained, we shall present the expectations.

6. Expectations vs. results

As we have seen dual superconductivity of the vacuum is not a well defined concept. There are infinitely many choices for the operator $\vec{\phi}(x)$, and for each of them there can or can not be condensation. What is the good choice, if any?

- A. There is club of practitioners of the “maximal abelian projection”, saying that their choice is better than others. In fact with this choice the abelian field “dominates” the configurations, in particular the part of it which is produced by monopoles. This can prove convenient to attempt a construction of effective lagrangeans, but in principle does not preclude any pattern of symmetry.
- B. There is a conjecture that all abelian projections are equivalent[14].

Most probably both attitudes reflect our imperfect knowledge of the symmetry of the disordered phase.

Discriminating between (A) and (B) is possible on the lattice. The results that we have obtained by a systematic study of dual superconductivity show unambiguously

- 1. That confinement is a transition from normal to dual superconductor ground state. This is strong evidence that the mechanism of confinement is indeed dual superconductivity of the vacuum.
- 2. A few different abelian projections have been analyzed. All of them show the same behaviour. The scenario (B) seems to be true.

This is reassuring for the validity of the mechanism itself. If only one abelian projection would show dual superconductivity only the particles with non zero charge with respect to that $U(1)$ could be confined: there exist states for which that charge is zero, e.g. the gluon which is parallel to $\hat{\phi}$. Moreover the colour direction of the electric field in the flux tubes observed in the lattice should also be parallel to $\hat{\phi}$, being the electric partner of the magnetic $U(1)$ of the monopoles. This has been shown to be not true[9,10]. Both these facts are naturally explained if the scenario (B) is at work.

We conclude by giving a few details on the technique used to detect dual superconductivity. The technique has been checked in many well known systems showing order disorder duality versus traditional descriptions[19].

7. The disorder parameter

An operator μ is constructed which carries non zero magnetic charge with respect to the $U(1)$ under study. Finite temperature is realized on the lattice by the usual thermodynamical recipe of having euclidean time running from 0 to $1/T$ (the temperature), with periodic boundary conditions for barions, antiperiodic for fermions. On a lattice this is done by using a size $N_s^3 \times N_t$, with $N_s \gg N_t$ and $a(\beta)N_t = 1/T$ ($\beta = 2N/g^2$). By renormalization group arguments $a(\beta) \simeq \frac{1}{\Lambda_L} \exp(-b_0\beta)$, or $T = \frac{\Lambda_L}{N_t} \exp(b_0\beta)$.

The technique used to construct μ is inspired to ref. [18] and [20] and is a complicated version of the simple formula for translations in elementary quantum mechanics

$$e^{ipa}|q\rangle = |q+a\rangle . \quad (28)$$

In the Schrödinger representation, the field $\vec{A}(x, t)$ plays the rôle of x , the conjugate momentum $\vec{\Pi}(\vec{x}, t)$ the rôle of p and

$$\mu|\vec{A}(\vec{x}, t)\rangle \equiv \exp\left(i \int d^3x \vec{\Pi}(\vec{x}, t) \cdot \vec{A}(\vec{x} - \vec{y})\right) |\vec{A}(\vec{x}, t)\rangle = |\vec{A}(\vec{x}, t) + \vec{A}(\vec{x} - \vec{y})\rangle . \quad (29)$$

If \vec{A} is the field of a monopole at \vec{y} , μ is indeed the creation operator for a monopole, at site \vec{y} and at time t . When inserted in the Feynman integral the operator is nothing but a linear term in the conjugated momentum added to the lagrangean and hence a shift of $\vec{\Pi}$ at time t . Care is needed to adapt the definition (29) to a compact system, in a form which does not depend on the choice of the gauge for the classical field \vec{A} . This can be done, and the result is, as sketched above[17,19]

$$\langle \mu \rangle = \frac{Z[S + \Delta S]}{Z[S]} , \quad (30)$$

with ΔS different from zero on a hyperplane at constant x_0 .

Being the exponential of a sum on N_s^3 sites $\langle \mu \rangle$ is subject to strong fluctuations. Numerically it is better to measure the quantity

$$\rho = \frac{d}{d\beta} \log \langle \mu \rangle = \langle S \rangle_S - \langle S + \Delta S \rangle_{S+\Delta S} \quad (31)$$

and to reconstruct $\langle \mu \rangle$ as

$$\langle \mu \rangle = \exp \left[\int_0^\beta \rho(\beta') d\beta' \right] . \quad (32)$$

A typical shape of ρ is shown in fig. 1, one of $\langle \mu \rangle$ in fig. 2. As is well known[21] $\langle \mu \rangle$ as an analytic function of β , can not vanish indetentically in the deconfined phase if the number of degrees of freedom is finite. An extrapolation to $N_s \rightarrow \infty$ must be done by finite size analysis. Fig. 3 shows that a few different abelian projections behave in the same way. Fig. 4 shows that two different monopole species of $SU(3)$ also behave in the same way.

As for the extrapolation to $N_s \rightarrow \infty$ three different ranges of β are considered:

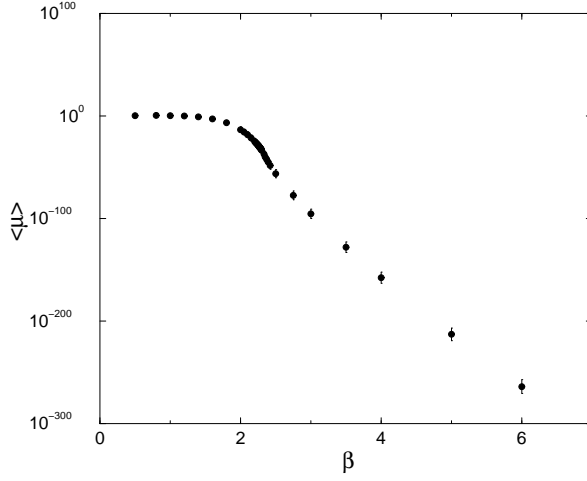


Figure 1. ρ vs. β for $SU(2)$ gauge theory. Plaquette projection, lattice $16^3 \times 4$.

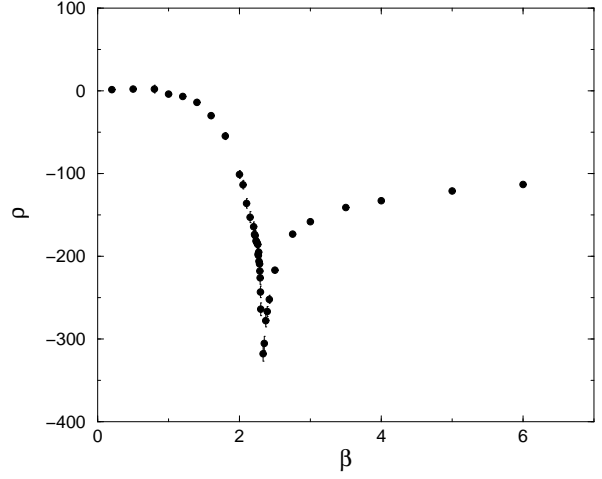


Figure 2. $\langle \mu \rangle$ vs. β for $SU(2)$ gauge theory. Plaquette projection, lattice $16^3 \times 4$.

1. $\beta \gg \beta_c$: there perturbative theory is at work and $\rho \simeq -cN_s + d$, with c a positive constant. As $N_s \rightarrow \infty$ $\rho \rightarrow -\infty$ and $\langle \mu \rangle = 0$.
2. $\beta \ll \beta_c$: there ρ has a finite limit as $N_s \rightarrow \infty$, and $\langle \mu \rangle$ is finite.
3. $\beta \simeq \beta_c$: there we expect $\langle \mu \rangle \simeq (\beta_c - \beta)^\delta$. From dimensional analysis

$$\langle \mu \rangle = N_s^\delta \Phi \left(\frac{a}{\xi}, \frac{\xi}{N_s}, \frac{N_t}{N_s} \right) . \quad (33)$$

Where $\xi \gg a$, and if $N_s \gg N_t$

$$\langle \mu \rangle = N_s^\delta \Phi \left(0, \frac{\xi}{N_s}, 0 \right) , \quad (34)$$

and since $\xi \simeq (\beta_c - \beta)^{-\nu}$, we get the scaling law

$$\frac{\rho}{N_s^{1/\nu}} = f \left(N_s^{1/\nu} (\beta_c - \beta) \right) . \quad (35)$$

The quality of the scaling is shown in fig. 5 for $SU(2)$ and in fig. 6 for $SU(3)$. It gives a determination of the critical indices ν , δ , and of β_c . For $SU(2)$ we get

$$\begin{aligned} \nu &= 0.63(5) \\ \beta_c &= 2.30(2) \quad N_t = 4 \quad . \\ \delta &= 0.20(8) \end{aligned}$$

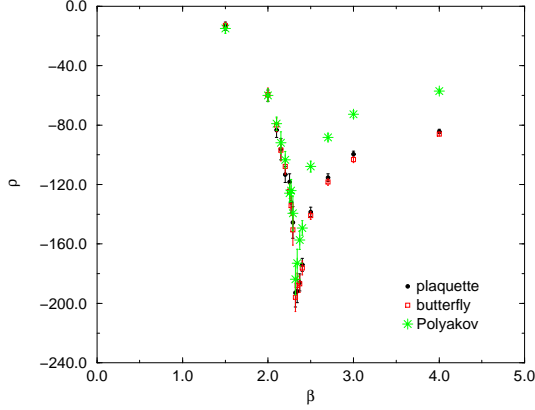


Figure 3. ρ vs. β for different abelian projections. $SU(2)$ gauge theory, lattice $12^3 \times 4$.

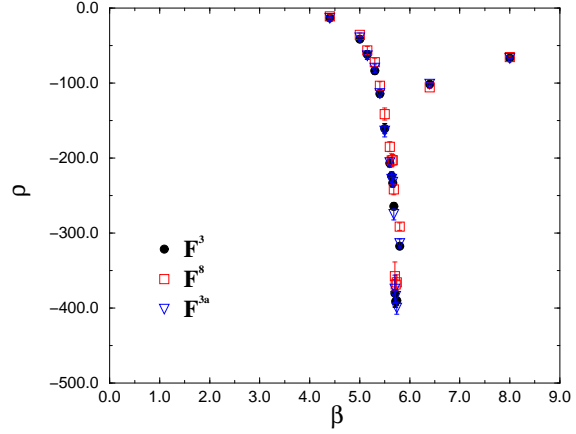


Figure 4. ρ vs. β for different abelian generators of $SU(3)$. Polyakov projection, lattice $12^3 \times 4$.

ν and β_c are in agreement within errors with independent determinations, and ν indicates a second order phase transition.

For $SU(3)$

$$\begin{aligned} \nu &= 0.33(2) \\ \beta_c &= 5.70(3) \quad N_t = 4 \quad , \\ \delta &= 0.54(4) \end{aligned}$$

indicating that the transition is first order.

The method used has been tested on a number of known systems and understood in its details. The result show beyond any reasonable doubt that dual superconductivity occurs, in different abelian projections, in connection with confinement.

The part of the work reported due to our group has been done in collaboration with L. Del Debbio, G. Paffuti, P. Pieri, B. Lucini, D. Martelli in the last few years. Their contribution was determinant to the results.

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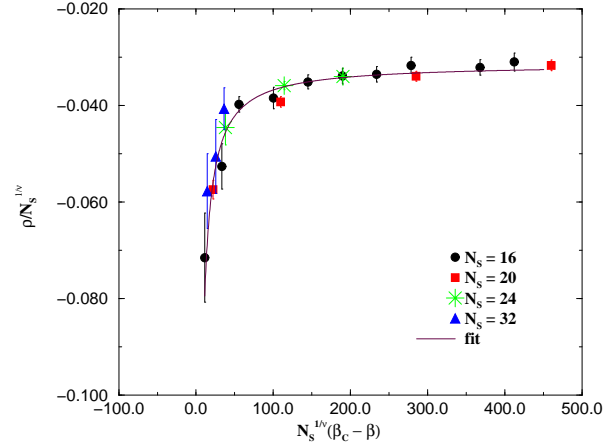
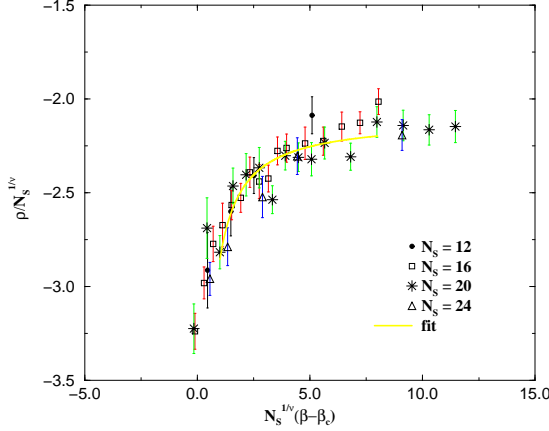


Figure 5. Quality of the scaling for $SU(2)$ gauge theory. Plaquette projection, $N_t = 4$. Figure 6. Quality of the scaling for $SU(3)$ gauge theory. Polyakov projection, $N_t = 4$.

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